AD-A066 210 FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/G 20/3

EXCITATION OF ELECTROMAGNETIC OSCILLATIONS IN OPEN RESONATORS,(U)

DEC 78 L A VAYNSHTEYN

UNCLASSIFIED FTD-ID(RS)T-2234-78 NL

1 OF 1

AD
A066210

END
DATE
FILMED
5--79
DDC

FTD-ID(RS)T-2234-78

FOREIGN TECHNOLOGY DIVISION

EXCITATION OF ELECTROMAGNETIC OSCILLATIONS
IN OPEN RESONATORS

By

L. A. Vaynshteyn

Approved for public release;
distribution unlimited.

EDITED TRANSLATION

FTD-ID(RS)T-2234-78 29 December 1978

MICROFICHE NR: FTD-79-C-000023

CSC65031231

EXCITATION OF ELECTROMAGNETIC OSCILLATIONS IN OPEN RESONATORS

By: L. A. Vaynshteyn

English pages: 29

Source: Elektronika Bol'shikh Moshchnostey, Issue 4, Moscow, 1965, pp. 157-172

Country of Origin: USSR
Translated by: Gale M. Weisenbarger
Requester: FTD/TQCS

U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

2234,gw

EXCITATION OF ELECTROMAGNETIC OSCILLATIONS IN OPEN RESONATORS

L. A. Vaynshteyn

An open resonator in the general case is a system of homogeneous or heterogeneous bodies (their properties are assigned by the distribution of complex permeabilities in space) which are surrounded by a vacuum. Oscillations in such a system are accompanied by radiation into free space. It is proposed that among the natural oscillations of the system there are natural oscillations with high quality with which we are familiar. The problem of induced oscillations of an open resonator under the effect of outside currents and the Cauchy problem are solved using expansion with respect to eigenfunctions of a continuous spectrum. In the solutions there is clear separation of the resonance part caused by natural oscillations with high quality. The general theory is illustrated

using the example of a homogeneous transparent sphere excited by a radial electric dipole.

INTRODUCTION

Open resonators are oscillatory systems whose oscillations are accompanied by radiation into free space. We define such a system assigning in space the distribution of complex permeabilities - dielectric and magnetic

(1) $$\varepsilon = \varepsilon(k) = \varepsilon(k; x, y, z), \ \mu = \mu(k) = \mu(k; x, y, z),$$

dependent on the wave number $k = \omega/c$ (or, what is the same thing, from the frequency ω) and from coordinates x, y, z whereby we shall consider that

(2) $$\varepsilon = \mu = 1 \text{ при } R = \sqrt{x^2 + y^2 + z^2} > \bar{R},$$

i.e., beyond the limits of a sphere with radius $\bar{R}$ - is a void in

which diverging electromagnetic waves are propagated which are formed during electromagnetic oscillations of the system. Complex permeabilities (1) correspond to absorbent or nonabsorbent substances, so that

$$\operatorname{Im} \varepsilon > 0, \ \operatorname{Im} \mu > 0. \tag{3}$$

We shall not examine active substances (with negative losses).

According to this definition an open resonator is a dielectric or metallic sphere just like the majority of other electrodynamic systems used in practice. The general theory presented in the article is applicable to a broad class of systems defined by conditions (1) - (3), however, it leads to simple results, which are of interest only in the case when the given system is a resonance system, i.e., sufficiently high-quality natural oscillations are possible in it (see §1). The latter requirement is satisfied by a dielectric sphere (see §5) while a metal sphere has practically no resonance properties (see [1], §6). Open resonators, formed by mirrors, placed in a vacuum, obviously are encompassed by the above definition: for them the theory which is presented below is a natural development of the theory given in [1] where we were limited by a scalar wave equation and ideally reflecting mirrors. Conventional cavity resonators become open resonators if they are connected with free space.

§1: ATTENUATING NATURAL OSCILLATIONS

Natural electromagnetic oscillations of a system are oscillations with which the electromagnetic field is dependent on time t according to the law

(4) $$\mathbf{E}(t) = \mathrm{Re}\{\mathbf{E}_s e^{-i\omega_s t}\},\ \mathbf{H}(t) = \mathrm{Re}\{\mathbf{H}_s e^{-i\omega_s t}\},$$

where

(5) $$\omega_s = \omega_s' - i\omega_s''$$

is a complex frequency of natural oscillation with subscript s.

Vector functions $\mathbf{E}_s = \mathbf{E}_s(x, y, z)$ and $\mathbf{H}_s = \mathbf{H}_s(x, y, z)$ satisfy Maxwell's uniform equations

(6) $$\mathrm{rot}\,\mathbf{E}_s = ik_s\mu\mathbf{H}_s,\ \mathrm{rot}\,\mathbf{H}_s = -ik_s\varepsilon\mathbf{E}_s\ \left(k_s = \frac{\omega_s}{c}\right).$$

Permeabilities ε and μ are determined by formulas (1)-(3) and

subsequently we shall take k at the frequency of excitation and only at the end of §4 will we need vectors $\mathbf{E}_s$ and $\mathbf{H}_s$, which satisfy equations (6) with $\varepsilon=\varepsilon(k_s)$ and $\mu=\mu(k_s)$, i.e., with permeabilities at the frequency of the natural oscillations themselves.

If and μ, as functions of x, y, z, undergo sudden changes then equations (6) are supplemented by boundary conditions on the interfaces. With R → ∞ vectors $\mathbf{E}_s$ and $\mathbf{H}_s$ satisfy the conditions of radiation

(7)
$$\mathbf{E}_s=\mathbf{g}_s(\vartheta,\varphi)\frac{e^{ik_sR}}{R},\ \mathbf{H}_s=[\mathbf{n}\mathbf{g}_s(\vartheta,\varphi)]\frac{e^{ik_sR}}{R},$$

where R,ϑ,φ are spherical coordinates; n is a single radial vector $(n_R=1,n_\vartheta=n_\varphi=0)$; $\mathbf{g}_s(\vartheta,\varphi)$ is a vector function which is single-valued on a single sphere and tangent to it so that $\mathbf{n}\mathbf{g}_s(\vartheta,\varphi)=0$.

Attenuation of natural oscillation in time (value ω_s'') serves as its measure) is determined both by radiation of a divergent wave (7) and by losses in the system itself (with complexity of or μ). The quality of oscillation

(8)
$$Q_s=\frac{\omega_s'}{2\omega_s''}$$

can be quite high (cf. §5).

§2. EIGENFUNCTIONS OF A CONTINUOUS SPECTRUM

Let us examine electromagnetic fields satisfying Maxwell's uniform equations (with $0<\varkappa<\infty$)

$$\mathrm{rot}\,\mathbf{E} = i\varkappa\mu\mathbf{H},\quad \mathrm{rot}\,\mathbf{H} = -i\varkappa\varepsilon\mathbf{E} \tag{9}$$

and having the following form with $R \to \infty$:

$$\left.\begin{aligned} \mathbf{E} &= \chi^{\circ}(\vartheta,\varphi)\frac{e^{-i\varkappa R}}{R} + \chi(\vartheta,\varphi)\frac{e^{i\varkappa R}}{R}, \\ \mathbf{H} &= -[\mathbf{n}\chi^{\circ}(\vartheta,\varphi)]\frac{e^{-i\varkappa R}}{R} + [\mathbf{n}\chi(\vartheta,\varphi)]\frac{e^{i\varkappa R}}{R}, \end{aligned}\right\} \tag{10}$$

where vectors χ° and χ are single-valued on a single sphere and are tangent to it. They are connected by the relationship

$$\chi = S\chi^{\circ}, \tag{11}$$

where S - the operator of scattering - is a linear integral operator,

dependent (with a fixed distribution of and μ in space) on parameter $\varkappa$. With $g_{\tau,\varkappa}(\vartheta,\varphi)$ let us designate the natural vector functions of the inverse operator S^{-1} and with $\Gamma_\tau(\varkappa)$, the corresponding eigenvalues. We have

(12) $$S^{-1}g_{\tau,\varkappa}=\Gamma_\tau(\varkappa)\,g_{\tau,\varkappa},$$

whre τ is the discrete subscript numbering the eigenfunctions with fixed $\varkappa$. For a sphere (see §5) subscript τ replaces symbols E_{mn} and H_{mn}, i.e., two regular subscripts m and n along with the indicated polarization.

Natural vector functions $E_{\tau,\varkappa}$ and $H_{\tau,\varkappa}$ of a continuous spectrum are introduced as a solutions of equations (9) having the following form with $R\to\infty$:

(13) $$\left.\begin{aligned}E_{\tau,\varkappa}&=g^{*}_{\tau,\varkappa}(\vartheta,\varphi)\frac{e^{-i\varkappa R}}{R}+g_{\tau,\varkappa}(\vartheta,\varphi)\frac{e^{i\varkappa R}}{R},\\ H_{\tau,\varkappa}&=-[ng^{*}_{\tau,\varkappa}(\vartheta,\varphi)]\frac{e^{-i\varkappa R}}{R}+[ng_{\tau,\varkappa}(\vartheta,\varphi)]\frac{e^{i\varkappa R}}{R},\end{aligned}\right\}$$

whereby on the strength of relationship (12)

(14) $$g^{*}_{\tau,\varkappa}(\vartheta,\varphi)=\Gamma_\tau(\varkappa)\,g_{\tau,\varkappa}(\vartheta,\varphi).$$

For functions $E_{\tau,\varkappa}$ and $H_{\tau,\varkappa}$ the condition of orthogonality is valid

$$\frac{1}{4\pi}\int \varepsilon E_{\tau,\varkappa}E_{\tau',\varkappa'}dV = -\frac{1}{4\pi}\int \mu H_{\tau,\varkappa}H_{\tau',\varkappa'}dV = D_\tau(\varkappa)\,\delta_{\tau\tau'}\delta(\varkappa-\varkappa'), \tag{15}$$

into which enters the product of the complex vector functions (without complex conjugation!). Under condition (15) integration is carried out through the entire infinite space, and the last integral is taken on the single sphere. Relationship (15) is derived from the identities

$$\left.\begin{aligned} \operatorname{div}[E_{\tau,\varkappa}H_{\tau',\varkappa'}] &= i\varkappa'\varepsilon E_{\tau,\varkappa}E_{\tau',\varkappa'} + i\varkappa\mu H_{\tau,\varkappa}H_{\tau',\varkappa'},\\ \operatorname{div}[E_{\tau',\varkappa'}H_{\tau,\varkappa}] &= i\varkappa\varepsilon E_{\tau,\varkappa}E_{\tau',\varkappa'} + i\varkappa'\mu H_{\tau,\varkappa}H_{\tau',\varkappa'} \end{aligned}\right\} \tag{18}$$

using the same arguments as in the case of the scalar wave equation (see [1], §2). Function (16) plays the role of the norm of eigenfunctions of the continuous spectrum.

Vector functions $E_{\tau,\varkappa}$ and $H_{\tau,\varkappa}$ may be continued analytically both on the negative part of the real axis and on the entire plane of the complex variable $\varkappa$. With negative $\varkappa$ they are determined by the

formulas

(19) $$E_{\tau,-\varkappa}=\epsilon_\tau E_{\tau,\varkappa},\ H_{\tau,-\varkappa}=-\epsilon_\tau H_{\tau,\varkappa},$$

where

(20) $$\epsilon_\tau=\pm 1.$$

With complex $\varkappa$ (namely in the lower half plane) the function $\Gamma_\tau(\varkappa)$ may vanish. Each root of the equation $\Gamma_\tau(\varkappa)=0$ coincides with one of the values k_s, examined in §1. In addition we have

(21) $$E_{\tau,\varkappa}=E_s,\ H_{\tau,\varkappa}=H_s\ \text{when}\ \varkappa=k_s.$$

Actually in this case according to formulas (13) and (14) there is one divergent spherical wave in accordance with formula (7). The norm of the attenuating fundamental oscillation with subscript s is the value

(22) $$N_s=\frac{i}{2\pi}\frac{dD_\tau(k_s)}{d\varkappa}=\frac{i}{4\pi}\frac{d\Gamma_\tau(k_s)}{d\varkappa}G_s,$$

where

(23) $$G_s = G_\gamma(k_s) = \int g_s^2 d\Omega,$$

Value N_s appears during calculation of remainders in the points $\varkappa = k_s$ (see §3 and §4). It is possible to prove (cf. [1], §3), that

(24) $$N_s = \lim_{R\to\infty e^{i\gamma}} \frac{1}{4\pi} \int\limits_{V_R} \varepsilon E_s^2 dV = -\lim_{R\to\infty e^{i\gamma}} \frac{1}{4\pi} \int\limits_{V_R} \mu H_s^2 dV,$$

where V_R is the volume of a sphere of radius R, and the angle γ is chosen so that the integrals converge. During approximate calculation of the norm it is sufficient to integrate through the finite range (cf. [1], §7).

§3. EXCITATION OF OPEN RESONATORS WITH ASSIGNED POINTS

Let the sources of the field be outside electrical and magnetic currents (with densities $\mathbf{j}^e$ and $\mathbf{j}^m$), which oscillate with a frequency of ω = ck and are located at finite distances. The field E, H excited by them must satisfy Maxwell's equations

(25) $$\mathrm{rot}\, \mathbf{E} = ik\mu \mathbf{H} - \frac{4\pi}{c} \mathbf{j}^m, \ \mathrm{rot}\, \mathbf{H} = -ik\varepsilon \mathbf{E} - \frac{4\pi}{c} \mathbf{j}^e$$

and the conditions for infinity

(26) $$\lim_{R\to\infty} RE = 0,\ \lim_{R\to\infty} RH = 0 \text{ при } \operatorname{Im} k > 0,$$

which provide singularity of the solution (see, for example, [2], §10) and which make it possible to look for it in the form of integrals (30) and (42) on the real axis. In the final formulas (43) and (44) it is possible to assume that Im k = 0.

Since vectors $E_{\tau,\varkappa}$ and $H_{\tau,\varkappa}$ satisfy relationships

$$\operatorname{div}(\varepsilon E_{\tau,\varkappa}) = 0,\ \operatorname{div}(\mu H_{\tau,\varkappa}) = 0,$$

(27) arising from equations (9), the sought fields E and H should be represented in the form (cf. [2], § 101 or [3])

(28) $$E = E' + E',\ H = H' + H',$$

where E', and H' are transverse (solenoidal) fields which satisfy the relationship

(29) $$\operatorname{div}(\varepsilon E') = 0,\ \operatorname{div}(\mu H') = 0$$

and may be represented in the form

(30) $$E' = \sum_\tau \int_0^\infty A_\tau(\varkappa) E_{\tau,\varkappa} d\varkappa,\ H' = \sum_\tau \int_0^\infty B_\tau(\varkappa) H_{\tau,\varkappa} d\varkappa,$$

and E^l, and H^l are longitudinal (potential) fields which may be written in the form

$$E^l = -\operatorname{grad}\Phi^e,\ H^l = -\operatorname{grad}\Phi^m. \tag{31}$$

Here Φ^e and Φ^m are the electrical and magnetic scalar potentials which satisfy the equations

$$\operatorname{div}(\varepsilon \operatorname{grad}\Phi^e) = -4\pi\rho^e,\ \operatorname{div}(\mu \operatorname{grad}\Phi^m) = -4\pi\rho^m, \tag{32}$$

where

$$\rho^e = -\frac{i}{ck}\operatorname{div} j^e,\ \rho^m = -\frac{i}{ck}\operatorname{div} j^m \tag{33}$$

are the densities of the outside charges.

Equations (32) are solved using electrostatic and magnetostatic methods (cf. §5). Longitudinal fields do not have resonance properties. Since all charges are at finite distances, Φ^e and Φ^m with $R \to \infty$ decrease like 1/R or faster. Longitudinal fields are orthogonal to the vector functions of the continuous spectrum:

(34) $$\int \varepsilon \mathbf{E}_{\tau,\varkappa} \mathbf{E}^{l} dV = 0, \quad \int \mu \mathbf{H}_{\tau,\varkappa} \mathbf{H}^{l} dV = 0.$$

Actually we may write

(35) $$\int\limits_{V_R} \varepsilon \mathbf{E}_{\tau,\varkappa} \mathbf{E}^{l} dV = -\int\limits_{V_R} \varepsilon \mathbf{E}_{\tau,\varkappa} \operatorname{grad} \Phi^{e} dV = -\int\limits_{V_R} \operatorname{div} (\varepsilon \mathbf{E}_{\tau,\varkappa} \Phi^{e}) dV + \\ +\int\limits_{V_R} \Phi^{e} \operatorname{div} (\varepsilon \mathbf{E}_{\tau,\varkappa}) dV = -\oint\limits_{S_R} \Phi^{e} \mathbf{n} \mathbf{E}_{\tau,\varkappa} dS,$$

and the integral over S_R (sphere of radius R) with $R \to \infty$ vanishes since Φ^{e} decreases as 1/R or faster and the radial component $E_{\tau,\varkappa}$, like $1/R^2$.

Substituting expansion (30) into equations (25) we obtain the relationships

(36) $$\left. \begin{aligned} i\varepsilon \sum_{\tau} \int_{0}^{\infty} \{kA_{\tau}(\varkappa) - \varkappa B_{\tau}(\varkappa)\} \mathbf{E}_{\tau,\varkappa} d\varkappa &= \frac{4\pi}{c} \mathbf{j}^{e} - ik\varepsilon \mathbf{E}^{l}, \\ i\mu \sum_{\tau} \int_{0}^{\infty} \{\varkappa A_{\tau}(\varkappa) - kB_{\tau}(\varkappa)\} \mathbf{H}_{\tau,\varkappa} d\varkappa &= -\frac{4\pi}{c} \mathbf{j}^{m} + ik\mu \mathbf{H}^{l}, \end{aligned} \right\}$$

in which solenoidal vectors stand on the right; for them it is possible to write the expansions

(37) $$\left. \begin{aligned} \frac{4\pi}{c} \mathbf{j}^{e} - ik\varepsilon \mathbf{E}^{l} &= \frac{\varepsilon}{c} \sum_{\tau} \int_{0}^{\infty} a_{\tau}(\varkappa) \mathbf{E}_{\tau,\varkappa} d\varkappa, \\ -\frac{4\pi}{c} \mathbf{j}^{m} + ik\mu \mathbf{H}^{l} &= \frac{\mu}{c} \sum_{\tau} \int_{0}^{\infty} b_{\tau}(\varkappa) \mathbf{H}_{\tau,\varkappa} d\varkappa, \end{aligned} \right\}$$

and coefficients $a_\tau(\varkappa)$ and $b_\tau(\varkappa)$ on the strength of formulas (15) and (34) are obtained in the form

(38)
$$a_\tau(\varkappa)=\frac{1}{D_\tau(\varkappa)}\int \mathbf{j}^e\mathbf{E}_{\tau,\varkappa}dV,\quad b_\tau(\varkappa)=\frac{1}{D_\tau(\varkappa)}\int \mathbf{j}^m\mathbf{H}_{\tau,\varkappa}dV.$$

Coefficients $A_\tau(\varkappa)$ and $B_\tau(\varkappa)$ must satisfy equations

(39)
$$\left.\begin{aligned} kA_\tau(\varkappa)-\varkappa B_\tau(\varkappa)&=-\frac{i}{c}a_\tau(\varkappa),\\ \varkappa A_\tau(\varkappa)-kB_\tau(\varkappa)&=-\frac{i}{c}b_\tau(\varkappa),\end{aligned}\right\}$$

hence

(40)
$$\left.\begin{aligned} A_\tau(\varkappa)&=-\frac{i}{c}\frac{ka_\tau(\varkappa)-\varkappa b_\tau(\varkappa)}{k^2-\varkappa^2},\\ B_\tau(\varkappa)&=-\frac{i}{c}\frac{\varkappa a_\tau(\varkappa)-kb_\tau(\varkappa)}{k^2-\varkappa^2}.\end{aligned}\right\}$$

Using relationships (19) and (20) and introducing coefficients

(41)
$$C_\tau(\varkappa)=-\frac{i}{2c}\frac{a_\tau(\varkappa)-b_\tau(\varkappa)}{k-\varkappa},$$

we may replace expansion (30) by the following:

(42) $$\mathbf{E}' = \sum_{\nu} \int_{-\infty}^{\infty} C_{\nu}(x)\,\mathbf{E}_{\nu,x}\,dx, \quad \mathbf{H}' = \sum_{\nu} \int_{-\infty}^{\infty} C_{\nu}(x)\,\mathbf{H}_{\nu,x}\,dx,$$

in which integration is performed along the entire real axis.

In these integrals let us displace the integration curve downward and designate with Γ the new curve, and by Δ the area between the old and new curves. We obtain

(43) $$\mathbf{E}' = \sum_{\Delta} C_s \mathbf{E}_s + \hat{\mathbf{E}}, \quad \mathbf{H}' = \sum_{\Delta} C_s \mathbf{H}_s + \hat{\mathbf{H}},$$

where summation is spread along all of the attenuating fundamental oscillations, the wave numbers k_s of which lie in the range Δ; $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ are represented by integrals (42) not on the real axis, but on curve Γ. Coefficients C_s on the strength of formulas (22) and (23) are obtained in the form

(44) $$C_s = -\frac{i}{2c}\,\frac{a_s - b_s}{k - k_s},$$

where

(45) $$a_s = \frac{1}{N_s}\int \mathbf{j}^e \mathbf{E}_s dV,\quad b_s = \frac{1}{N_s}\int \mathbf{j}^m \mathbf{H}_s dV.$$

With proper selection of the range Δ (see [1], §4) the sums in formulas (43) determine the resonance part of the field and against the background of which appear resonance properties of the given system. It may happen that the given system does not have resonance properties either in general or with a certain arrangement of the sources or observation points: then the separation of the resonance part does not make sense (see [1], §4).

The formulas derived above are in many ways analogous to the formulas which are obtained in the theory of excitation of "closed" cavity resonators (see [2], [3]).

§4. CAUCHY PROBLEM FOR ELECTROMAGNETIC OSCILLATIONS

If (formally) we count all currents of conductivity among currents of displacement then Maxwell's equations for nonstationary fields can be written in the form

(46) $$\operatorname{rot}\mathbf{E}(t) = -\frac{1}{c}\frac{\partial \mathbf{B}(t)}{\partial t},\quad \operatorname{rot}\mathbf{H}(t) = \frac{1}{c}\frac{\partial \mathbf{D}(t)}{\partial t}.$$

The Cauchy problem reduces to the integration of these equations under the initial conditions

$$\mathbf{D}(t) = \mathbf{D}^{\circ},\ \mathbf{B}(t) = \mathbf{B}^{\circ} \text{ при } t = 0, \tag{47}$$

where D^{0} and B^{0} are assigned vector functions which decrease sufficiently rapidly with $R \to \infty$.

Let us introduce E(k), the vector function of coordinates x, y, z, according to the formulas

$$\mathbf{E}(k) = \int_{0}^{\infty} e^{ikct}\mathbf{E}(t)\,dt,\ \mathbf{E}(t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} e^{-ikct}\mathbf{E}(k)\,dk \tag{48}$$

and analogously determine H(k), D(k), and B(k). From equations (40) we obtained for these vectors the equations

$$\operatorname{rot}\mathbf{E}(k) = ik\mathbf{B}(k) + \frac{1}{c}\mathbf{B}^{\circ},\ \operatorname{rot}\mathbf{H}(k) = -ik\mathbf{D}(k) - \frac{1}{c}\mathbf{D}^{\circ}. \tag{49}$$

If we consider the relationships

(50) $$\mathbf{D}(k)=\varepsilon(k)\mathbf{E}(k),\quad \mathbf{B}(k)=\mu(k)\mathbf{H}(k),$$

in which $\varepsilon(k)$ and $\mu(k)$ are the same as in formulas (1), then equations (49) are reduced to equations (25) with outside currents

(51) $$\mathbf{j}^e=-\frac{1}{4\pi}\mathbf{D}^\circ,\quad \mathbf{j}^m=-\frac{1}{4\pi}\mathbf{B}^\circ.$$

The conditions

(52) $$\operatorname{div}\mathbf{D}^\circ=0,\quad \operatorname{div}\mathbf{B}^\circ=0$$

ensure the absence of longitudinal fields so that the complete field $\mathbf{E}(k)$, $\mathbf{H}(k)$ is obtained in the form of integrals (30) and (42). However, for writing these integrals and for formulating the conditions (26) which ensure singularity of the solution, in formulas (48) it is necesary to consider that $\mathrm{Im}\, k > 0$ and to integrate by k somewhat above the real axis.

Separating the resonance part into $\mathbf{E}(k)$ and $\mathbf{H}(k)$ according to formula (43) and selecting range Δ in the form of a band

(53) $$-\delta<\mathrm{Im}\,\varkappa<0,\quad -\infty<\mathrm{Re}\,\varkappa<\infty,$$

we make a transition to E(t) and H(t) using the second formula (48). Shifting the integration curve downward toward the straight line Im $\varkappa$ = $-\delta$, we obtain the expression

$$\mathbf{E}(t)=\sum_{\Delta}C_s(t)\mathbf{E}_s+\ldots,\quad \mathbf{H}(t)=\sum_{\Delta}C_s(t)\mathbf{H}_s+\ldots, \tag{54}$$

in which the terms are explicity written which decrease with t → ∞ with respect to absolute value more slowly than $e^{-\delta ct}$, and moreover

$$\left.\begin{aligned}&C_s(t)=-\frac{1}{2c}(a_s-b_s)\,e^{-ik_sct},\\&a_s=-\frac{1}{4\pi N_s}\int \mathbf{D}^\circ\mathbf{E}_s\,dV,\quad b_s=-\frac{1}{4\pi N_s}\int \mathbf{B}^\circ\mathbf{H}_s\,dV.\end{aligned}\right\} \tag{55}$$

In formulas (54) and (55) figure weakly attenuating oscillations with a frequency $\omega_s=ck_s$ ($\mathrm{Im}\,k_s>-\delta$), which satisfy equations (6) with $\varepsilon=\varepsilon(k_s)$ and $\mu=\mu(k_s)$. If the substances filling or forming the open resonator themselves possess resonance properties, thanks to which there appears a strong dependence of ε and μ on k then these oscillations can differ considerably from the oscillations examined earlier. Even the number of these oscillations may be different. Let, for example, the working part of the resonator (let us say, the space between the mirrors) be filled with a homogeneous substance in which

(56) $$\varepsilon(k)=\varepsilon_0-\frac{A_0}{k^2+2ip_0k-k_0^2},\quad \mu(k)=1,$$

where ε_0, A_0, and p_0 do not depend on frequency. If $k_s\approx k_0$, then during replacement of $\varepsilon(k)$ by $\varepsilon(k_s)$ splitting of frequency ck_s into two frequencies which correspond to various oscillations is possible.

Natural values k_s are distributed in the plane of a complex variable symmetrically relative to the imaginary axis so that to frequency ω_s and to vectors E_s, H_s always correspond frequency $-\omega_s^*$ and vectors E_s^*, H_s^*: this follows from expressions (4). In the problem of excitation of monochromatic fields (§3) values k_s, which lie near the negative part of the real axis are not of interest; in the Cauchy problem they ensure reality of the sums (54).

Derivation of formulas (43)-(45) is based on the supposition (cf. [1], end of §4) that integrands (42) in the range Δ except for simple bands k_s do not have any other special features. In §5 we examine the behavior of these functions in a particular case of a homogeneous isotropic sphere and are convinced that this is actually so. During derivation of formulas (54) and (55) we further proposed that the essentially singular points of functions $\varepsilon(k)$ and $\mu(k)$ lie below the straight line $\mathrm{Im}x = -\delta$, i.e., that δ is sufficiently small. Bands of the functions $\varepsilon(k)$ and $\mu(k)$, for example in the case of applicability of formulas (56) the points $\pm\sqrt{k^2-p_0^2}-ip_0$, may lie

above this straight line (i.e, it may be considered tha $p_0 < \delta$).

§5. HOMOGENEOUS ISOTROPIC SPHERE AS AN OPEN RESONATOR

Let a homogeneous sphere of radius a with permeabilities and μ be excited by the radial electric dipole located in point $R = R_0$, $\vartheta = 0$. Outside the sphere is a vacuum ($\varepsilon = \mu = 1$ with $R > a$), the dipole may be outside the sphere as well as inside, the moment of the dipole p, the frequency $\omega = ck$. The induced field is expressed, as is known, by the scalar function $U = U(R, \vartheta)$:

$$\left.\begin{aligned} &E_R = -\frac{1}{R^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right), \quad E_\vartheta = \frac{1}{R}\frac{\partial^2 U}{\partial R\,\partial\vartheta}, \\ &E_\varphi = H_R = H_\vartheta = 0, \quad H_\varphi = \frac{ik\varepsilon}{R}\frac{\partial U}{\partial\vartheta}. \end{aligned}\right\} \tag{57}$$

The eigenfunctions of the continuous spectrum which we need are equal to

$$\left.\begin{aligned} &U_{\text{on},\varkappa}(R, \vartheta) = \\ &= [B_n^{(1)}(\varkappa)\, h_n^{(1)}(\varkappa R) - B_n^{(2)}(\varkappa)\, h_n^{(2)}(\varkappa R)]\, P_n(\cos\vartheta) \quad \text{при } R > a, \\ &U_{\text{on},\varkappa}(R, \vartheta) = \frac{2i}{\varepsilon}\, j_n(\varkappa\sqrt{\varepsilon\mu}\, R)\, P_n(\cos\vartheta) \qquad \text{при } R < a, \end{aligned}\right\} \tag{58}$$

where j_n, $h_n^{(1)}$ and $h_n^{(2)}$ are connected with the Bessel and Hunkel functions

(59) $$j_n(x) = \sqrt{\frac{\pi x}{2}} J_{n+\frac{1}{2}}(x), \quad h_n^{(1,2)}(x) = \sqrt{\frac{\pi x}{2}} H^{(1,2)}_{n+\frac{1}{2}}(x),$$

and

(60) $$\left.\begin{aligned} B_n^{(1)}(\varkappa) &= \sqrt{\frac{\mu}{\varepsilon}}\, j_n'(\varkappa\sqrt{\varepsilon\mu}\,a)\, h_n^{(2)}(\varkappa a) - j_n(\varkappa\sqrt{\varepsilon\mu}\,a)\, h_n^{(2)\prime}(\varkappa a), \\ B_n^{(2)}(\varkappa) &= \sqrt{\frac{\mu}{\varepsilon}}\, j_n'(\varkappa\sqrt{\varepsilon\mu}\,a)\, h_n^{(1)}(\varkappa a) - j_n(\varkappa\sqrt{\varepsilon\mu}\,a)\, h_n^{(1)\prime}(\varkappa a). \end{aligned}\right\}$$

Substituting functions $U_{0n,\varkappa}$ into formulas (57) where k is replaced by $\varkappa$, we find the vector functions $E_{0n,\varkappa}$ and $H_{0n,\varkappa}$. Functions $U_{mn,\varkappa}$ we obtain by replacing Legendre's polynomial $P_n(\cos\vartheta)$ by $P_n^m(\cos\vartheta)^{\cos m\varphi}_{\sin m\varphi}$, in formulas (57), however, in our problem they are not necessary.

The solution of the posed problem has the form

(61) $$U = \sum_{n=0}^{\infty} U_n P_n(\cos\vartheta), \quad U_n = U_n^t + U_n^l,$$

where U_n^t corresponds to the transverse, and U_n^l to the longitudinal electrical field. We find values U_n^l solving the electrostatic problem. With $R_0 > a$ we have

$$(62)\quad \left.\begin{aligned} U_n^l &= p\frac{2n+1}{(\varepsilon+1)n+1}\frac{R^{n+1}}{R_0^{n+2}} && \text{with при } R<a,\\ U_n^l &= p\left[1+\frac{(\varepsilon-1)(n+1)}{(\varepsilon+1)n+1}\left(\frac{a}{R}\right)^{2n+1}\right]\frac{R^{n+1}}{R_0^{n+2}} && \text{with при } a<R<R_0,\\ U_n^l &= p\left[1+\frac{(\varepsilon-1)(n+1)}{(\varepsilon+1)n+1}\left(\frac{a}{R_0}\right)^{2n+1}\right]\frac{R_0^{n-1}}{R^n} && \text{with при } R>R_0, \end{aligned}\right\}$$

and with $R_0 < a$

$$(63)\quad \left.\begin{aligned} U_n^l &= \frac{p}{\varepsilon}\left[1-\frac{(\varepsilon-1)n}{(\varepsilon+1)n+1}\left(\frac{R_0}{a}\right)^{2n+1}\right]\frac{R^{n+1}}{R_0^{n+2}} && \text{with при } R<R_0,\\ U_n^l &= \frac{p}{\varepsilon}\left[1-\frac{(\varepsilon-1)n}{(\varepsilon+1)n+1}\left(\frac{R}{a}\right)^{2n+1}\right]\frac{R_0^{n-1}}{R^n} && \text{with при } R_0<R<a,\\ U_n^l &= p\frac{2n+1}{(\varepsilon+1)n+1}\frac{R_0^{n-1}}{R^n} && \text{with при } R>a. \end{aligned}\right\}$$

Values U_n^l we find from the general theory (§3) replacing the dipole with a concentrated electrical current and using the formulas

$$(64)\quad \left.\begin{aligned} \Gamma_{0n}(\varkappa) &= (-1)^{n+1}\frac{B_n^{(2)}(\varkappa)}{B_n^{(1)}(\varkappa)},\\ G_{0n}(\varkappa) &= 4\pi(-1)^n\frac{n(n+1)}{2n+1}\varkappa^2[B_n^{(1)}(\varkappa)]^2,\\ D_{0n}(\varkappa) &= -2\pi\frac{n(n+1)}{2n+1}\varkappa^2 B_n^{(1)}(\varkappa)B_n^{(2)}(\varkappa). \end{aligned}\right\}$$

Using the identity $P_n(1)=1$, the result may be written in the form

$$(65)\quad U_n^l = \frac{kp}{4\pi R_0^2}(2n+1)\int_{-\infty}^{\infty}\frac{U_{0n,\varkappa}(R_0,0)\,U_{0n,\varkappa}(R,0)}{(k-\varkappa)\varkappa^2 B_n^{(1)}(\varkappa)B_n^{(2)}(\varkappa)}\,d\varkappa.$$

The correctness of this formula is not difficult to check supplementing it with the term U_n^l and using the theorem of remainders; with $R_0 > R > a$ we obtain

(66)
$$U_n = \frac{i\rho}{2kR_0^2}(2n+1)h_n^{(1)}(kR_0)\left[h_n^{(2)}(kR) - \frac{B_n^{(1)}(k)}{B_n^{(2)}(k)}h_n^{(1)}(kR)\right],$$

and formulas (57) and (61) give us the solution in the classical form.

Integral (65) may be converted by deforming the integration curve downward. Integrand (65) is meromorphic which is easy to perceive from the properties of functions (59). In the point $\varkappa = 0$ it is regular. In the lower half plane $\mathrm{Im}\,\varkappa < 0$ its bands coincide with the roots of the equation $B_n^{(2)}(\varkappa) = 0$, which can be written in the form

(67)
$$\sqrt{\frac{\mu}{\varepsilon}}\frac{j_n'(\varkappa\sqrt{\varepsilon\mu}\,a)}{j_n(\varkappa\sqrt{\varepsilon\mu}\,a)} = \frac{h_n^{(1)\prime}(\varkappa a)}{h_n^{(1)}(\varkappa a)}.$$

The possibility of representing integral (65) in the form of a sum of remainders in points k_{nq}, the roots of equation (67), is determined only by the behavior of the integrand with $\mathrm{Im}\,\varkappa \to -\infty$.

Let us limit ourselves to real positive values $\sqrt{\varepsilon\mu}$. With $R_0 > a$ and $R > a$ the integrand in the lower half plane increases as $e^{-i\varkappa(R_0+R-2a)}$, therefore the integral on the closing semicircle $|\varkappa| = K$ with $K \to \infty$ does not vanish (cf. [1], §6) and it is not possible to

reduce the integral to only the remainders. With $R_0 < a$ and $R < a$ this reduction is always possible. With $R_0 > a$ and $R < a$ the integral is reduced to remainders under the condition

$$R_0 - a < \sqrt{\varepsilon\mu}(a - R). \tag{66}$$

However, the possibility of reducing integral (65) to a sum of remainders is not connected directly with the presence of resonance properties of the sphere. The latter are determined by the existence of bands k_{nq} with a sufficiently small imaginary part and also by the existence of nonintersecting resonance curves, i.e., by the disposition of bands k_{nq} in range Δ, adjacent to the real axis (see §3).

Let us examine the roots of equation (67) with $\varkappa a \gg 1$ and $\sqrt{\varepsilon\mu} > 1$, when the beams inside the sphere can experience total reflection from its boundary. Thanks to the permeation of energy into the surrounding space the fundamental oscillations corresponding to such beams will attenuate. It is apparent that the smallest radiation attenuations will be possessed by oscillations having the character of waves of a whispering gallery, for which the angle of incidence of beams on the boundary of the sphere is close to $\pi/2$. For such oscillations functions $j_n'(x)$ and $j_n(x)$ may be replaced by Fok's asymptotic expressions (see, for example, [4]), which are suitable with $x \approx n + 1/2$ and

functions $h_n^{(1)'}(x)$ и $h_n^{(1)}(x)$ by Debay's asymptotic expressions, which are suitable with x < n + 1/2. Equation (67) then takes the form

$$\frac{v'(t)}{v(t)} = \sqrt{\frac{\varepsilon}{\mu}}\,\nu\,\mathrm{sh}\,\eta\,(1 - ie^{-2T}), \tag{69}$$

where

$$\left.\begin{aligned} t &= \frac{n+\frac{1}{2} - \varkappa\sqrt{\varepsilon\mu}\,a}{\nu}, \quad \nu = \left(\frac{n+\frac{1}{2}}{2}\right)^{\frac{1}{3}}, \\ \mathrm{ch}\,\eta &= \frac{n+\frac{1}{2}}{\varkappa a}, \quad T = \left(n+\frac{1}{2}\right)(\eta - \mathrm{th}\,\eta). \end{aligned}\right\} \tag{70}$$

Considering parameter ν to be large it is possible to represent t in the form $t_q^{\circ} + \Delta t_{nq}$, where $t_q^{\circ} - q$ is the root of the equation

$$v(t) = 0 \quad (q = 1, 2, \ldots; \; t_q^{\circ} < 0), \tag{71}$$

and value Δt_{nq} is equal to

$$\left.\begin{aligned} \Delta t_{nq} &= \frac{1 + ie^{-2T_{nq}}}{\nu\sqrt{\frac{\varepsilon}{\mu}(\varepsilon\mu - 1)}}, \\ T_{nq} &= \left(n+\frac{1}{2}\right)\left(\mathrm{Arch}\sqrt{\varepsilon\mu} - \sqrt{1-\frac{1}{\varepsilon\mu}}\right) + \nu t_q^{\circ}\sqrt{1-\frac{1}{\varepsilon\mu}} + \frac{1}{\varepsilon}. \end{aligned}\right\} \tag{72}$$

Hence

$$k_{nq} = \frac{n+\frac{1}{2} - \nu(t_q^{\circ} + \Delta t_{nq})}{\sqrt{\varepsilon\mu}\,a}, \tag{73}$$

so that the radiation quality is equal to

$$Q_{nq} = \frac{n+\frac{1}{2}}{2}\sqrt{\frac{\varepsilon}{\mu}(\varepsilon\mu-1)}\,e^{2T_{nq}}. \tag{74}$$

It increases with an increase in subscript n and decreases with an increase of subscript q.

Formulas (67) -(74) relate to electrical oscillations E_{mnq} of a homogeneous isotropic sphere. If in these formulas ε and μ change places then we will obtain formulas for magnetic oscillations H_{mnq}. The resonance properties of spherical particles are displayed during scattering of electromagnetic waves on them (see, for example, [5]). In [6] light was generated in a spherical-shaped crystal; oscillations E_{mnq} were induced - waves of a whispering gallery, with 2n + 1 - the multiple confluence of eigenvalues (73) led to the simultaneous generation of many oscillations. However, the predominant excitation of oscillations with small q is connected not with formula (74), which gives an extremely high quality, but with the fact that for them, according to the known criterion of Rayleigh (see, for example, [7]), the roughness of the boundary is less pronounced. Let us note that the formula given in [6] for quality,

similar to formula (74) is erroneous.

CONCLUSION

This work examined the problem of excitation of an open resonance system by outside currents and also the Cauchy problem. Formal solutions of these problems were constructed using expansion of the sought fields with respect to eigenfunctions of a continuous spectrum. For the resonance part of the field simple expressions are derived which make it possible to calculate it if the weakly attenuating fundamental oscillations are known which determine it. These expressions make it possible to calculate a number of properties of open resonators approximately the same way as this is done for conventional cavity resonators.

I would like to take this opportunity to thank P. L. Kapitsa and V. A. Fok for their interest in this work.

LITERATURE

1. Л. А. Вайнштейн. Возбуждение открытых резонаторов, ЖТФ, 34, № 9, 1541, 1964.
2. Л. А. Вайнштейн. Электромагнитные волны, гл. XVII. М., изд-во «Советское радио», 1957.
3. Л. А. Вайнштейн. ЖТФ, 23, № 4, 646, 1953.

4. В. А. Фок. Дифракция радиоволн вокруг земной поверхности. М.—Л Изд-во АН СССР, 1946, стр. 55.

5. Г. ван де Х ю л с т. Рассеяние света малыми частицами. Изд-во иностр. лит-ры, 1961, стр. 179—186.

6. C. G. B. Garret, W. Kaiser, W. L. Bond. Phys. Rev., 124, № 6, 1807, 1961. Сборник переводов «Лазеры» под ред. М. Е. Жаботинского. Изд-во иностр. лит-ры, 1963, стр. 408.

7. P. Beckmann, A. Spizzichino. The scattering of electromagnetic waves from rough surfaces. Pergamon Press, Oxford, 1963 (Chapter 2).

DISTRIBUTION LIST

DISTRIBUTION DIRECT TO RECIPIENT

ORGANIZATION		MICROFICHE
A205	DMATC	1
A210	DMAAC	2
B344	DIA/RDS-3C	9
C043	USAMIIA	1
C509	BALLISTIC RES LABS	1
C510	AIR MOBILITY R&D LAB/FIO	1
C513	PICATINNY ARSENAL	1
C535	AVIATION SYS COMD	1
C591	FSTC	5
C619	MIA REDSTONE	1
D008	NISC	1
H300	USAICE (USAREUR)	1
P005	DOE	1
P050	CIA/CRB/ADD/SD	1
NAVORDSTA (50L)		1
NASA/KSI		1
AFIT/LD		1
LLL/Code L-389		1

ORGANIZATION		MICROFICHE
E053	AF/INAKA	1
E017	AF/RDXTR-W	1
E403	AFSC/INA	1
E404	AEDC	1
E408	AFWL	1
E410	ADTC	1
	FTD	
	CCN	1
	ASD/FTD/NIIS	3
	NIA/PHS	1
	NIIS	2